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## FRACTIONAL CALCULUS OPERATOR AND ITS APPLICATIONS IN THE UNIVALENT FUNCTIONS

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**ABSTRACT.** In this paper we introduce the class  $\mathcal{A}(\alpha, \beta, \gamma)$  consisting of analytic functions which is defined by using the fractional calculus operator  $\mathcal{J}_z^\lambda$  in the unit disk  $\mathcal{U}$ . We shall determine the relationships of this class and well known classes  $\mathcal{S}^*(\gamma)$  and  $\mathcal{K}(\gamma)$  and investigate coefficient estimates and growth theorems for functions belonging to the class  $\mathcal{A}(\alpha, \beta, \gamma)$ . Integral operator  $F_c$  is also considered for the class  $\mathcal{A}(\alpha, \beta, \gamma)$ .

### 1. Introduction and Definitions

Let  $\mathcal{A}$  denote the class of functions of the form

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disk  $\mathcal{U} = \{z : |z| < 1\}$ . Also let  $\mathcal{S}$  denote the class of all functions in  $\mathcal{A}$  which are univalent in the unit disk  $\mathcal{U}$ .

A function  $f(z)$  belonging to the class  $\mathcal{S}$  is said to be starlike of order  $\gamma$  ( $0 \leq \gamma < 1$ ) if and only if

$$(1.2) \quad \operatorname{Re} \left( \frac{zf'(z)}{f(z)} \right) > \gamma \quad (z \in \mathcal{U}; 0 \leq \gamma < 1).$$

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We denote by  $\mathcal{S}^*(\gamma)$  the subclass of  $\mathcal{S}$  consisting of functions which are starlike of order  $\gamma$  in  $\mathcal{U}$ .

Further, a function  $f(z)$  belonging to the class  $\mathcal{S}$  is said to be convex of order  $\gamma$  ( $0 \leq \gamma < 1$ ) if and only if

$$(1.3) \quad \operatorname{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \gamma \quad (z \in \mathcal{U}; 0 \leq \gamma < 1).$$

We denote by  $\mathcal{K}(\gamma)$  the subclass of  $\mathcal{S}$  consisting of functions which are convex of order  $\gamma$  in  $\mathcal{U}$ .

We note that

$$(1.4) \quad f(z) \in \mathcal{K}(\gamma) \Leftrightarrow zf'(z) \in \mathcal{S}^*(\gamma),$$

and that  $\mathcal{S}^*(\gamma) \subset \mathcal{S}^*(0) \equiv \mathcal{S}^*$  and  $\mathcal{K}(\gamma) \subset \mathcal{K}(0) \equiv \mathcal{K}$  ( $0 \leq \gamma < 1$ ), where  $\mathcal{S}^*$  and  $\mathcal{K}$  denote the subclasses of  $\mathcal{A}$  consisting of functions which are starlike and convex in  $\mathcal{U}$ , respectively.

Many essentially equivalent definitions of fractional calculus have been given in the literature (*cf.*, *e.g.*, [6] and [7, p.45]). We state the following definitions due to Owa and Srivastava [5] which have been used rather frequently in the theory of analytic functions (see also [3]).

**Definition 1.** The fractional integral of order  $\lambda$  is defined, for a function  $f(z)$ , by

$$(1.5) \quad \mathcal{D}_z^{-\lambda} f(z) := \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta \quad (\lambda > 0),$$

and the fractional derivative of order  $\lambda$  is defined, for a function  $f(z)$ , by

$$(1.6) \quad \mathcal{D}_z^\lambda f(z) := \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\lambda} d\zeta \quad (0 \leq \lambda < 1),$$

where  $f(z)$  is an analytic function in a simply-connected region of the  $z$ -plane containing the origin, and the multiplicity of  $(z-\zeta)^{\lambda-1}$  involved in (1.5) (and that of  $(z-\zeta)^{-\lambda}$  involved in (1.6)) is removed by requiring  $\log(z-\zeta)$  to be real when  $z-\zeta > 0$ .

**Definition 2.** Under the hypotheses of Definition 1, the fractional derivative of order  $n + \lambda$  is defined by

$$(1.7) \quad \mathcal{D}_z^{n+\lambda} f(z) := \frac{d^n}{dz^n} \mathcal{D}_z^\lambda f(z) \quad (0 \leq \lambda < 1; n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}).$$

With the aid of the above definitions, Owa and Srivastava [5] defined the fractional operator  $\mathcal{J}_z^\lambda$  by

$$(1.8) \quad \mathcal{J}_z^\lambda f(z) = \Gamma(2-\lambda) z^\lambda \mathcal{D}_z^\lambda f(z) \quad (\lambda \neq 2, 3, 4, \dots)$$

for functions (1.1) belonging to the class  $\mathcal{A}$ .

We introduce the class  $\mathcal{A}(\alpha, \beta, \gamma)$  of analytic functions  $f(z)$  belonging to  $\mathcal{A}$  satisfying the condition

$$(1.9) \quad \operatorname{Re} \left( \frac{\mathcal{J}_z^\alpha f(z)}{\mathcal{J}_z^\beta f(z)} \right) > \gamma \quad (z \in \mathcal{U}).$$

for  $\alpha < 2$ ,  $\beta < 2$  and  $\gamma < 1$ .

We note that  $\mathcal{A}(1, 0, \gamma) = \mathcal{S}^*(\gamma)$  and  $\mathcal{A}(\alpha + 1, 0, \gamma) = \mathcal{S}^*(\gamma, \alpha)$  which was studied by Owa and Shen [4]. Also, for  $\lambda < 1$  and  $-\lambda/(1 - \lambda) \leq \gamma < 1$ ,  $\mathcal{A}(\lambda + 1, \lambda, \gamma) = \mathcal{V}(2, 2 - \lambda, (1 - \lambda)\gamma + \lambda)$ , which was studied by Kim and Srivastava [3].

In this paper, we find coefficient estimates and growth theorems for analytic functions belonging to the class  $\mathcal{A}(\alpha, \beta, \gamma)$  associated with the fractional calculus operator. We also point out the relationships between the class  $\mathcal{A}(\alpha, \beta, \gamma)$  and  $\mathcal{S}^*(\gamma)$  (or  $\mathcal{K}(\gamma)$ ).

## 2. Preliminary Results

In order to establish our results, we need the following lemmas.

**Lemma 1.** *Let the function  $f(z)$  is defined by (1.1) and let  $\lambda < 1$ . Then*

$$(2.1) \quad z(\mathcal{J}_z^\lambda f(z))' = (1 - \lambda)\mathcal{J}_z^{\lambda+1} f(z) + \lambda\mathcal{J}_z^\lambda f(z) \quad (z \in \mathcal{U}).$$

*Proof.* Using the definition of fractional calculus, we have

$$(2.2) \quad \mathcal{J}_z^\lambda f(z) = z + \sum_{n=2}^{\infty} \phi(n, \lambda) a_n z^n,$$

where

$$(2.3) \quad \phi(n, \lambda) = \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} \quad (n \geq 2).$$

By applying (2.2), we obtain

$$\begin{aligned} z(\mathcal{J}_z^\lambda f(z))' &= z + \sum_{n=2}^{\infty} n\phi(n, \lambda) a_n z^n \\ &= (1 - \lambda)\{z + \sum_{n=2}^{\infty} \phi(n, \lambda + 1) a_n z^n\} + \lambda\{z + \sum_{n=2}^{\infty} \phi(n, \lambda) a_n z^n\} \end{aligned}$$

which completes the proof of Lemma 1.

**Lemma 2.** (Jack [2]) *Let  $\omega(z)$  be analytic in  $\mathcal{U}$  with  $\omega(0) = 0$ . Then if  $|\omega(z)|$  attains its maximum value on the circle  $|z| = r$  at a point  $z_0$ , we can write*

$$z_0 \omega'(z_0) = k \omega(z_0),$$

where  $k$  is real and  $k \geq 1$ .

**Lemma 3.** (Srivastava and Owa [8]) *If the function  $f(z)$  defined by (1.1) satisfies  $\operatorname{Re}(f(z)/z) > \delta$  ( $0 \leq \delta < 1$ ), then*

$$(2.4) \quad \sum_{n=2}^{\infty} |a_n| \leq 1 - \delta.$$

The result (2.4) is sharp.

**Lemma 4.** (Twomey [10]) *Let the function  $f(z)$  defined by (1.1) be in the class  $S^*$ . Then*

$$(2.5) \quad \left| \frac{z f'(z)}{f(z)} \right| \leq 1 + \frac{|z| \ln \left( \frac{(1+|z|)^2 |f(z)|}{|z|} \right)}{(1-|z|) \ln \left( \frac{1+|z|}{1-|z|} \right)} \quad (z \in \mathcal{U}).$$

Equality in (2.5) holds true for the Koebe function  $\kappa(z) = z/(1-z)^2$ .

### 3. Main Results

We begin by proving

**Theorem 1.** *Let  $\alpha < 2$ ,  $\beta < 2$  and  $\gamma < 1$ . If  $f(z) \in \mathcal{A}(\alpha, \beta, \gamma)$ , then*

$$(3.1) \quad |a_n| \leq \frac{2(1-\gamma)}{|\phi(n, \alpha) - \phi(n, \beta)|} \prod_{j=2}^{n-1} \left( 1 + \frac{2(1-\gamma)\phi(j, \beta)}{|\phi(j, \alpha) - \phi(j, \beta)|} \right) \quad (n \geq 2),$$

where  $\phi(n, \alpha)$  and  $\phi(n, \beta)$  are given by (2.3). The result is sharp.

*Proof.* If we set

$$(3.2) \quad p(z) = \frac{\frac{\mathcal{J}_z^\alpha f(z)}{\mathcal{J}_z^\beta f(z)} - \gamma}{1 - \gamma} = 1 + c_1 z + c_2 z^2 + \cdots \quad (f \in \mathcal{A}),$$

then  $p(z)$  is analytic with  $p(0) = 1$  and has positive real part in  $\mathcal{U}$ . Since  $\mathcal{J}_z^\alpha f(z) = ((1-\gamma)p(z) + \gamma)\mathcal{J}_z^\beta f(z)$ , by virtue of (2.2), we have

$$(\phi(n, \alpha) - \phi(n, \beta)) a_n = (1 - \gamma) \left\{ c_{n-1} + \sum_{m=2}^{n-1} \phi(m, \beta) c_{n-m} a_m \right\} \quad (n \geq 2).$$

By applying Carathéodory's Lemma (see [1, p.41]), we obtain

$$(3.3) \quad |\phi(n, \alpha) - \phi(n, \beta)| |a_n| \leq 2(1 - \gamma) \left\{ 1 + \sum_{m=2}^{n-1} \phi(m, \beta) |a_m| \right\}.$$

We will prove, using mathematical induction, that the assertion (3.1) is satisfied for  $n \geq 2$ . If  $n = 2$ , then

$$|a_2| \leq \frac{2(1 - \gamma)}{|\phi(2, \alpha) - \phi(2, \beta)|}.$$

Now suppose that the assertion (3.1) is satisfied for  $n \leq k$ . Then, from (3.1) and (3.3) we have

$$\begin{aligned} & |\phi(k+1, \alpha) - \phi(k+1, \beta)| |a_{k+1}| \\ & \leq 2(1 - \gamma) \left\{ 1 + \sum_{m=2}^k \phi(m, \beta) |a_m| \right\} \\ & \leq 2(1 - \gamma) \left\{ 1 + \sum_{m=2}^k \phi(m, \beta) \frac{2(1 - \gamma)}{|\phi(m, \alpha) - \phi(m, \beta)|} \prod_{j=2}^{m-1} \left( 1 + \frac{2(1 - \gamma)\phi(j, \beta)}{|\phi(j, \alpha) - \phi(j, \beta)|} \right) \right\} \\ & = 2(1 - \gamma) \prod_{j=2}^k \left( 1 + \frac{2(1 - \gamma)\phi(j, \beta)}{|\phi(j, \alpha) - \phi(j, \beta)|} \right). \end{aligned}$$

Hence

$$|a_n| \leq \frac{2(1 - \gamma)}{|\phi(n, \alpha) - \phi(n, \beta)|} \prod_{j=2}^{n-1} \left( 1 + \frac{2(1 - \gamma)\phi(j, \beta)}{|\phi(j, \alpha) - \phi(j, \beta)|} \right)$$

for all  $n \geq 2$ .

Finally, the result is sharp for the function  $f(z)$  given by

$$f(z) = z + \frac{2(1 - \gamma)}{|\phi(n, \alpha) - \phi(n, \beta)|} \prod_{j=2}^{n-1} \left( 1 + \frac{2(1 - \gamma)\phi(j, \beta)}{|\phi(j, \alpha) - \phi(j, \beta)|} \right) z^n \quad (n \geq 2).$$

**Remark 1.** Letting  $\alpha = 1$ ,  $\beta = 0$  and  $\gamma = 0$  in Theorem 1, we immediately obtain that

$$f(z) \in \mathcal{S}^* \Rightarrow |a_n| \leq n$$

for all  $n \geq 2$  ([1, Theorem 2.14]).

**Theorem 2.** Let  $\lambda < 1$  and  $-\lambda/(1-\lambda) \leq \gamma < 1$ . Then  $f(z) \in \mathcal{A}(\lambda+1, \lambda, \gamma)$  if and only if  $\mathcal{J}_z^\lambda f(z) \in \mathcal{S}^*((1-\lambda)\gamma + \lambda)$ .

*Proof.* In view of Lemma 1, we have

$$(3.4) \quad \frac{z(\mathcal{J}_z^\lambda f(z))'}{\mathcal{J}_z^\lambda f(z)} = (1-\lambda) \frac{\mathcal{J}_z^{\lambda+1} f(z)}{\mathcal{J}_z^\lambda f(z)} + \lambda.$$

Assume that  $f(z) \in \mathcal{A}(\lambda+1, \lambda, \gamma)$ . Then, from (3.4) we obtain

$$\begin{aligned} \operatorname{Re} \left( \frac{z(\mathcal{J}_z^\lambda f(z))'}{\mathcal{J}_z^\lambda f(z)} \right) &= (1-\lambda) \operatorname{Re} \left( \frac{\mathcal{J}_z^{\lambda+1} f(z)}{\mathcal{J}_z^\lambda f(z)} \right) + \lambda \\ &> (1-\lambda)\gamma + \lambda. \end{aligned}$$

Thus  $\mathcal{J}_z^\lambda f(z) \in \mathcal{S}^*((1-\lambda)\gamma + \lambda)$ .

Conversely, suppose that  $\mathcal{J}_z^\lambda f(z) \in \mathcal{S}^*((1-\lambda)\gamma + \lambda)$ . In view of Lemma 1 and (3.4), it is clear that

$$\operatorname{Re} \left( \frac{\mathcal{J}_z^{\lambda+1} f(z)}{\mathcal{J}_z^\lambda f(z)} \right) > \gamma.$$

This completes the proof of Theorem 2.

By virtue of Theorem 2 and Lemma 4, we obtain

**Corollary 1.** Let  $\lambda < 1$  and  $-\lambda/(1-\lambda) \leq \gamma < 1$ . Then  $zf'(z) \in \mathcal{A}(\lambda+1, \lambda, \gamma)$  if and only if  $\mathcal{J}_z^\lambda f(z) \in \mathcal{K}((1-\lambda)\gamma + \lambda)$ .

*Proof.* By (1.8) and Theorem 2, it follows that

$$z(\mathcal{J}_z^\lambda f(z))' = \mathcal{J}_z^\lambda (zf'(z)) \in \mathcal{S}^*((1-\lambda)\gamma + \lambda).$$

Hence, from (1.4) we obtain  $\mathcal{J}_z^\lambda f(z) \in \mathcal{K}((1-\lambda)\gamma + \lambda)$ .

**Corollary 2.** Let  $\lambda < 1$  and let  $f(z) \in \mathcal{A}(\lambda+1, \lambda, -\lambda/(1-\lambda))$ . Then

$$(3.5) \quad \left| \frac{z(\mathcal{J}_z^\lambda f(z))'}{\mathcal{J}_z^\lambda f(z)} \right| \leq 1 + \frac{|z| \ln \left( \frac{(1+|z|)^2 |\mathcal{J}_z^\lambda f(z)|}{|z|} \right)}{(1-|z|) \ln \left( \frac{1+|z|}{1-|z|} \right)} \quad (z \in \mathcal{U}).$$

Equality in (3.5) holds true for the function  $f(z) = h(z) * (z/(1-z)^2)$ , where

$$h(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+\lambda-1)}{\Gamma(2-\lambda)n!} z^n \quad (z \in \mathcal{U})$$

and the operator  $*$  stands for the Hadamard product or convolution of two regular functions.

With the aid of Lemma 1 and Lemma 2, we prove

**Theorem 3.** Let  $\lambda < 1$  and  $0 \leq \delta < 1$ . Suppose that  $f(z) \in \mathcal{A}(\lambda + 1, \lambda, \gamma(\delta))$ , where

$$\gamma(\delta) = \begin{cases} 1 - \frac{\delta}{2(1-\lambda)(1-\delta)} & (0 \leq \delta \leq \frac{1}{2}) \\ 1 - \frac{1-\delta}{2\delta(1-\lambda)} & (\frac{1}{2} \leq \delta < 1). \end{cases}$$

Then

$$(3.6) \quad \operatorname{Re} \left( \frac{\mathcal{J}_z^\lambda f(z)}{z} \right) > \delta \quad (z \in \mathcal{U}).$$

*Proof.* If we define the function  $\omega$  by

$$(3.7) \quad \frac{\mathcal{J}_z^\lambda f(z)}{z} = \frac{1 + (2\delta - 1)\omega(z)}{1 + \omega(z)} \quad (z \in \mathcal{U}),$$

then  $\omega$  is analytic in  $\mathcal{U}$  with  $\omega(0) = 0$  and  $\omega(z) \neq -1$ . Making use of the logarithmic differentiation of both side in (3.7), we have

$$\frac{z(\mathcal{J}_z^\lambda f(z))'}{\mathcal{J}_z^\lambda f(z)} = 1 - \frac{2(1-\delta)z\omega'(z)}{(1+\omega(z))(1+(2\delta-1)\omega(z))}.$$

By Lemma 1, we obtain

$$\frac{\mathcal{J}_z^{\lambda+1} f(z)}{\mathcal{J}_z^\lambda f(z)} = 1 - \frac{2(1-\delta)}{1-\lambda} \frac{z\omega'(z)}{(1+\omega(z))(1+(2\delta-1)\omega(z))}.$$

Suppose that there exists a point  $z_0 \in \mathcal{U}$  such that  $\max_{|z| \leq |z_0|} |\omega(z)| = |\omega(z_0)| = 1$ . Then, by using Lemma 2, we get

$$\operatorname{Re} \left( \frac{\mathcal{J}_z^{\lambda+1} f(z_0)}{\mathcal{J}_z^\lambda f(z_0)} \right) = 1 - \frac{2k(1-\delta)}{1-\lambda} \frac{\delta}{|1 + (2\delta - 1)\omega(z_0)|^2}.$$

When  $0 \leq \delta \leq 1/2$ ,

$$\operatorname{Re} \left( \frac{\mathcal{J}_z^{\lambda+1} f(z_0)}{\mathcal{J}_z^\lambda f(z_0)} \right) \leq 1 - \frac{\delta}{2(1-\lambda)(1-\delta)}.$$

When  $1/2 \leq \delta < 1$ ,

$$\operatorname{Re} \left( \frac{\mathcal{J}_z^{\lambda+1} f(z_0)}{\mathcal{J}_z^\lambda f(z_0)} \right) \leq 1 - \frac{1-\delta}{2\delta(1-\lambda)}.$$

These contradict the hypothesis that  $f(z) \in \mathcal{A}(\lambda + 1, \lambda, \gamma(\delta))$ . Hence  $|\omega(z)| < 1$  for all  $z \in \mathcal{U}$ . Thus, from (3.7) we obtain the desired result.

Setting  $\delta = 1/2$  in Theorem 3, we have



**Corollary 3.** *Let  $\lambda < 1$ . If  $f(z) \in \mathcal{A}(\lambda + 1, \lambda, (1 - 2\lambda)/2(1 - \lambda))$ , then*

$$\operatorname{Re} \left( \frac{\mathcal{J}_z^\lambda f(z)}{z} \right) > \frac{1}{2} \quad (z \in \mathcal{U}).$$

**Remark 2.** Taking  $\lambda = 0$  in Corollary 3, we see that  $f(z) \in \mathcal{S}^*(1/2)$  implies  $\operatorname{Re}(f(z)/z) > 1/2$ . Since  $\mathcal{K} \subset \mathcal{S}^*(1/2)$ , Corollary 3 is a generalization of the result due to Stroh  cker [9] (see also Duren [1, p.72]).

Next, by using Lemma 3 and Theorem 3, we have

**Corollary 4.** *Under the hypotheses of Theorem 3, let the function  $f(z)$  is defined by (1.1). Then*

$$(3.8) \quad |z| - (1 - \delta)|z|^2 \leq |\mathcal{J}_z^\lambda f(z)| \leq |z| + (1 - \delta)|z|^2$$

for  $z \in \mathcal{U}$ . Equality in all cases occurs for the function

$$(3.9) \quad f(z) = z + \frac{(2 - \lambda)(1 - \delta)}{2} z^2 \exp(i\theta)$$

at  $z = \pm|z| \exp(-i\theta)$ .

*Proof.* Notice from (2.2), (3.6) and Lemma 3 that

$$(3.10) \quad \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} |a_n| \leq 1 - \delta.$$

By using (2.2) and (3.10), we have

$$(3.11) \quad \begin{aligned} |\mathcal{J}_z^\lambda f(z)| &\geq |z| - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} |a_n| |z|^n \\ &\geq |z| - |z|^2 \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} |a_n| \\ &\geq |z| - (1 - \delta)|z|^2 \end{aligned}$$

and

$$(3.12) \quad \begin{aligned} |\mathcal{J}_z^\lambda f(z)| &\leq |z| + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} |a_n| |z|^n \\ &\leq |z| + (1 - \delta)|z|^2 \end{aligned}$$

for  $z \in \mathcal{U}$ . Combining the inequalities in (3.11) and (3.12), we obtain Corollary 4.

**Corollary 5.** Under the hypotheses of Theorem 3, let the function  $f(z)$  is defined by (1.1) and let  $0 \leq \lambda < 1$ . Then

$$(3.13) \quad |z| - \frac{(2-\lambda)(1-\delta)}{2}|z|^2 \leq |f(z)| \leq |z| + \frac{(2-\lambda)(1-\delta)}{2}|z|^2$$

for  $z \in \mathcal{U}$ . This result is sharp with an extremal function  $f(z)$  given by (3.9).

*Proof.* Observing that  $\phi(n, \lambda)$  given by (2.3) is non-decreasing of  $n$  for fixed  $\lambda$  ( $0 \leq \lambda < 1$ ), we find from (3.10) that

$$\sum_{n=2}^{\infty} |a_n| \leq \frac{(2-\lambda)(1-\delta)}{2}.$$

Hence, by using the same technique as detailed in the proof of Corollary 4, we obtain the assertion (3.13) of Corollary 5.

Finally, we state and prove

**Theorem 4.** Let  $0 \leq \delta < 1$ ,  $c \geq -\delta$ ,  $c^2 + 2\delta(1+c) \geq 1$  and  $1/(2(c+\delta)+1) \leq \gamma < 1$ . If  $f \in \mathcal{A}((\delta-2\gamma+1)/(1-\gamma), (\delta-\gamma)/(1-\gamma), \gamma - (1-\gamma)/2(c+\delta))$ , then the function  $F_c(z)$ , defined by

$$(3.14) \quad F_c(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt \quad (f \in \mathcal{A}; z \in \mathcal{U})$$

belongs to  $\mathcal{A}((\delta-2\gamma+1)/(1-\gamma), (\delta-\gamma)/(1-\gamma), \gamma)$ .

*Proof.* Let  $\lambda = (\delta-\gamma)/(1-\gamma)$ . From (3.14), we obtain

$$(3.15) \quad z (\mathcal{J}_z^\lambda F_c(z))' + c \mathcal{J}_z^\lambda F_c(z) = (c+1) \mathcal{J}_z^\lambda f(z).$$

Define the function  $\omega(z)$  by

$$(3.16) \quad \frac{z (\mathcal{J}_z^\lambda F_c(z))'}{\mathcal{J}_z^\lambda F_c(z)} = \frac{1 + (2\delta-1)\omega(z)}{1 + \omega(z)} \quad (0 \leq \delta < 1; z \in \mathcal{U}).$$

Here  $\omega(z)$  is analytic in  $\mathcal{U}$  with  $\omega(0) = 0$  and  $\omega(z) \neq -1$ . In view of (3.15), the assertion (3.16) yields

$$(3.17) \quad \frac{\mathcal{J}_z^\lambda f(z)}{\mathcal{J}_z^\lambda F_c(z)} = \frac{(1+c) + (2\delta-1+c)\omega(z)}{(1+c)(1+\omega(z))}.$$

Differentiating both side of (3.17) logarithmically, it follows that

$$(3.18) \quad \frac{z (\mathcal{J}_z^\lambda f(z))'}{\mathcal{J}_z^\lambda f(z)} = \delta + (1-\delta) \frac{1-\omega(z)}{1+\omega(z)} - \frac{2(1-\delta)z\omega'(z)}{(1+\omega(z))(1+c+(2\delta-1+c)\omega(z))}.$$

By assuming  $\max_{|z| \leq |z_0|} |\omega(z)| = |\omega(z_0)| = 1$  for  $z_0 \in \mathcal{U}$  and using the same technique as in the proof of Theorem 3, we find that (3.18) yields

$$\begin{aligned} \operatorname{Re} \left( \frac{z_0 (\mathcal{J}_z^\lambda f(z_0))'}{\mathcal{J}_z^\lambda f(z_0)} \right) &= \delta - \frac{2k(1-\delta)(c+\delta)}{|1+c+(2\delta-1+c)\omega(z_0)|^2} \\ &\leq \delta - \frac{1-\delta}{2(c+\delta)}. \end{aligned}$$

This contradicts the assumption that  $f(z) \in \mathcal{A}(\lambda+1, \lambda, \gamma - (1-\gamma)/2(c+\delta))$ , that is,

$$\operatorname{Re} \left( \frac{z \mathcal{J}_z^{\lambda+1} f(z)}{\mathcal{J}_z^\lambda f(z)} \right) = \frac{1}{(1-\lambda)} \left[ \operatorname{Re} \left( \frac{z (\mathcal{J}_z^\lambda f(z))'}{\mathcal{J}_z^\lambda f(z)} \right) - \lambda \right] > \gamma - \frac{1-\gamma}{2(c+\delta)}$$

for  $\lambda = (\delta - \gamma)/(1 - \gamma)$ . Therefore  $\omega(z)$  has to satisfy that  $|\omega(z)| < 1$  for all  $z \in \mathcal{U}$ . Hence, by (3.16) and Theorem 2,  $\mathcal{J}_z^\lambda F_c(z) \in \mathcal{S}^*(\delta)$  and  $F_c(z) \in \mathcal{A}(\lambda+1, \lambda, \gamma)$  for  $\lambda = (\delta - \gamma)/(1 - \gamma)$ .

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